

Supersymmetric Three-Form Flux Perturbations on AdS_5

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Abstract

We consider warped type IIB supergravity solutions with three-form flux and $\mathcal{N} = 1$ supersymmetry, which arise as the supergravity duals of confining gauge theories. We first work in a perturbation expansion around $AdS_5 \times S^5$, as in the work of Polchinski and Strassler, and from the $\mathcal{N} = 1$ conditions and the Bianchi identities recover their first-order solution generalized to an arbitrary $\mathcal{N} = 1$ superpotential. We find the second order dilaton and axion by the same means. We also find a simple family of exact solutions, which can be obtained from solutions found by Becker and Becker, and which includes the recent Klebanov–Strassler solution.

1 Introduction

The duality proposed by Maldacena [1] between type IIB string theory in $AdS_5 \times S^5$ and $\mathcal{N} = 4$ Yang Mills in four dimensions has been an arena for amazing theoretical advances in the past few years. This duality applies to a conformal four dimensional theory, and so as it stands it cannot shed light on confining gauge theories such as QCD. However, from it one can deduce dualities involving systems with less supersymmetry, including confining gauge theories.

In addition to their relation to gauge theories, deformations of AdS_5 with reduced supersymmetry are of interest for their connection to the proposal of Randall and Sundrum [2, 3]. These authors suggested that the hierarchy problem could be solved in a higher dimensional space with a large warp factor. Such warped spaces can be realized in string theory in various ways, most directly by bringing together N D3-branes on a Calabi-Yau manifold to produce a region that is locally $AdS_5 \times S^5$ [4]. In the simplest case there is $\mathcal{N} = 4$ supersymmetry, but one is ultimately interested in at most $\mathcal{N} = 1$.

One means of reducing the supersymmetry is by perturbing the Hamiltonian, which corresponds to perturbing the boundary conditions on $AdS_5 \times S^5$ [5, 6]. The breaking of $\mathcal{N} = 4$ to $\mathcal{N} = 1$ by mass terms has been studied from various points of view [7, 8, 9, 10, 11, 12]. The supergravity dual to this perturbed theory appears to contain a naked singularity, but in a recent paper [13] (henceforth PS), Polchinski and Strassler showed that this singularity is actually replaced by an expanded brane source, so the theory is tractable.

The supersymmetry can also be reduced by placing D3-branes at a singular point, such as an orbifold or conifold point [14, 15, 16, 17], rather than a regular point. To break the conformal invariance it is necessary in addition to introduce fractional branes [18, 19]. Again the supergravity dual appears at first to be singular [20], but recent work by Klebanov and Strassler (henceforth KS) has shown that the true solution is nonsingular [21]. Somewhat surprisingly, explicit branes are not involved; rather, the distinctive feature of the nonsingular solution is a reduced (spontaneously broken) symmetry.

In this paper we explore warped IIB supergravity solutions with unbroken $\mathcal{N} = 1$ supersymmetry. Our initial goal is to understand the supersymmetry of the PS solution. The $\mathcal{N} = 1$ mass in the gauge theory is dual to a three-form flux in the supergravity solution; we work in the same approximation as in PS, treat this flux as a perturbation.

In section 2 we review the type IIB supergravity fields and supersymmetry variations, and the zeroth order IIB solution that corresponds to the Coulomb branch of the $\mathcal{N} = 4$ gauge theory. In section 3 we first solve the conditions for unbroken $\mathcal{N} = 1$ supersymmetry

at first order around a general Coulomb branch solution. We then impose the Bianchi identities and find that the general solution is characterized by one holomorphic function ϕ and one harmonic function ψ . We verify that this general solution to the supersymmetry and Bianchi conditions also satisfies the equations of motion. The holomorphic function corresponds to an arbitrary $\mathcal{N} = 1$ superpotential, while the harmonic function corresponds to a higher dimension perturbation.

The KS solution also involves a three-form flux, but this cannot be regarded as a perturbation. In section 4 we note a simple class of exact solutions, which includes the KS solution and its $\mathcal{N} = 2$ version [22]. This class is actually a special case of a class of M/F theory solutions found by Becker and Becker [23, 24, 25].

In section 5 we make concluding remarks. An appendix contains various extensions of the work in section 3: the second order solutions for the dilaton and axion (which can be obtained easily because they decouple from the other second order perturbations); a simple particular solution; and, a verification that the solution obtained here agrees with that found in PS (in particular, that the normalizable solution is determined in terms of the warp factor).

2 Review of IIB supergravity

2.1 Fields and variations

The massless bosonic fields of the type IIB superstring theory consist of the dilaton Φ , the metric tensor g_{MN} and the antisymmetric 2-tensor B_{MN} in the NS-NS sector, and the axion C , the 2-form potential C_{MN} and the four-form field C_{MNPQ} with self-dual five-form field strength in the R-R sector. Their fermionic superpartners are a complex Weyl gravitino ψ_M ($\gamma^{11}\psi_M = -\psi_M$) and a complex Weyl dilatino λ ($\gamma^{11}\lambda = \lambda$). The theory has $\mathcal{N}=2$ supersymmetry generated by two supercharges of the same chirality. The two scalars can be combined into a complex field $\tau = C + ie^{-\phi}$ that parameterizes the $SL(2, \mathbf{R})/U(1)$ coset space.

We want to find background that preserve some supersymmetry. Assuming that the background fermi fields vanish, we have to find a combination of the bosonic fields such that the supersymmetry variation of the fermionic fields is zero. The equations for the variation of the dilatino and gravitino have been found in [26], whose conventions we use. We use subindices $M, N, \dots = 0, \dots, 9$; $\mu, \nu = 0, 1, 2, 3$ and $m, n, \dots = 4, \dots, 9$

$$\delta\lambda = \frac{i}{\kappa}\gamma^M P_M \varepsilon^* - \frac{i}{24}\gamma^{MNP} G_{MNP} \varepsilon, \quad (2.1)$$

$$\delta\psi_M = \frac{1}{\kappa}(D_M - \frac{1}{2}iQ_M)\varepsilon + \frac{i}{480}\gamma^{M_1\dots M_5}F_{M_1\dots M_5}\varepsilon + \frac{1}{96}\left(\gamma_M^{PQR}G_{PQR} - 9\gamma^{PQ}G_{MPQ}\right)\varepsilon^* . \quad (2.2)$$

Here

$$P_M = f^2\partial_MB , \quad Q_M = f^2\text{Im}(B\partial_MB^*) , \quad (2.3)$$

$$B = \frac{1+i\tau}{1-i\tau} , \quad f^{-2} = 1 - BB^* . \quad (2.4)$$

The supersymmetry parameter ε is a complex Weyl spinor ($\gamma^{11}\varepsilon = -\varepsilon$), and D_M is the covariant derivative with respect to the metric g_{MN} . The field strengths are

$$G_{(3)} = f(F_{(3)} - BF_{(3)}^*) , \quad F_{(3)} = dA_{(2)} , \quad (2.5)$$

$$F_{(5)} = dA_{(4)} - \frac{\kappa}{8}\text{Im}(A_{(2)} \wedge F_{(3)}^*) . \quad (2.6)$$

with $A_{(2)} = C_{(2)} + iB_{(2)}$ complex and $A_{(4)}$ real.

2.2 Black 3-brane background

For any distribution of D3-branes aligned along the μ -directions, the background is

$$ds^2 = Z^{-1/2}\eta_{\mu\nu}dx^\mu dx^\nu + Z^{1/2}dx^m dx^m , \quad (2.7)$$

$$F_{\mu\nu\lambda\rho m} = \frac{1}{4\kappa Z}\epsilon_{[\mu\nu\lambda\rho}\partial_m]Z , \quad F_{mnpqr} = -\frac{1}{4\kappa Z}\epsilon_{mnpqrs}\partial^s Z , \quad (2.8)$$

with τ constant and $G_{(3)} = 0$. For N D3-branes at $x^m = 0$,

$$Z = \frac{R^4}{r^4} , \quad R^4 = 4\pi g N \alpha'^2 \quad (2.9)$$

where $r^2 = x^m x^m$; the spacetime is then $AdS_5 \times S^5$.¹ More generally, Z is any function of the x^m , and

$$-\partial_m \partial_m Z(x^m) = (2\pi)^4 \alpha'^2 g \rho_3(x^m) \quad (2.10)$$

where $\rho_3(x^m)$ is the density of D3-branes in the transverse space.

For this background the dilatino equation (2.1) is automatically satisfied. The gravitino variations (2.2) in this background are

$$\kappa\delta\psi_\mu = \partial_\mu\varepsilon - \frac{1}{8}\gamma_\mu\gamma_w(1 - \Gamma^4)\varepsilon , \quad (2.11)$$

$$\kappa\delta\psi_m = \partial_m\varepsilon + \frac{1}{8}\varepsilon w_m - \frac{1}{8}\gamma_w\gamma_m(1 - \Gamma^4)\varepsilon , \quad (2.12)$$

¹Note that κ and g are related, $2\kappa^2 = (2\pi)^7 \alpha'^4 g^2$.

where $\Gamma^4 = i\gamma^{0123}$ is the four-dimensional chirality and we have defined

$$w_m = \partial_m \ln Z, \quad \gamma_w = \gamma^m w_m. \quad (2.13)$$

The spin connection has been calculated for tangent space axes \hat{M} parallel to the Cartesian coordinate axes M . The Poincaré supersymmetries are independent of x^μ and so the vanishing of $\delta\psi_\mu$ implies that

$$\Gamma^4 \varepsilon = \varepsilon. \quad (2.14)$$

The vanishing of $\delta\psi_m$ then implies that

$$\varepsilon = Z^{-1/8} \eta. \quad (2.15)$$

where η is any constant spinor such that $\Gamma^4 \eta = \eta$. We can decompose

$$\eta = \zeta \otimes \chi \quad (2.16)$$

where ζ is any four-dimensional anticommuting spinor with positive Γ^4 -chirality and χ is any six-dimensional commuting spinor with negative Γ^6 -chirality. There are two such ζ and four such χ , giving eight complex or 16 real supersymmetries.

3 Perturbative solutions

3.1 Supersymmetry conditions

The PS solution [13] was obtained in a perturbation expansion in powers of $G_{(3)}$ around the D3-brane solution (2.7,2.8). This expansion was justified in PS because it was found, in large regions of parameter space, that the D3-brane density dominated the density of the 5-branes that appeared in the resolution of the naked singularity. We can expand the supersymmetry parameter in the same way,

$$\varepsilon = \varepsilon_0 + \varepsilon_1 + \dots, \quad (3.1)$$

where as above

$$\varepsilon_0 = Z^{-1/8} \eta = Z^{-1/8} \zeta \otimes \chi. \quad (3.2)$$

We are looking for solutions with $D = 4$, $\mathcal{N} = 1$ supersymmetry so only one choice of the spinor χ is kept.²

²IIB backgrounds with $\mathcal{N} = 1$ supersymmetry were studied in refs. [27, 28], but with the assumption that the transverse dimensions are compact and without brane or other sources.

It is convenient to adopt complex coordinates z^i for the 6-dimensional part of the metric:

$$z^1 = \frac{x^4 + ix^7}{\sqrt{2}} , \quad z^2 = \frac{x^5 + ix^8}{\sqrt{2}} , \quad z^3 = \frac{x^6 + ix^9}{\sqrt{2}} . \quad (3.3)$$

In these coordinates, the six-dimensional part of the metric (2.7) is

$$g_{i\bar{j}} = Z^{1/2} \delta_{i\bar{j}} . \quad (3.4)$$

By an $SO(6)$ rotation we can choose η such that

$$\gamma_i \varepsilon_0 = \gamma^{\bar{i}} \varepsilon_0 = 0 . \quad (3.5)$$

We now expand the supersymmetry equations in powers of $G_{(3)}$. In the PS solution, the dilaton, metric, and five-form receive no first order correction so the first order dilatino equation is just

$$G \varepsilon_0 = 0 \quad (3.6)$$

where $G = G_{MNP} \gamma^{MNP} = G_{mnp} \gamma^{mnp}$. Expanding in complex coordinates and using the property (3.5), this is

$$0 = G_{ijk} \gamma^{ijk} \varepsilon_0 + 3 G_{i\bar{j}k} \gamma^{\bar{i}jk} \varepsilon_0 = G_{ijk} \gamma^{ijk} \varepsilon_0 + 6 G^j_{jk} \gamma^k \varepsilon_0 \quad (3.7)$$

The spinors in the two terms are independent, so

$$G_{ijk} = G^j_{jk} = 0 . \quad (3.8)$$

The first order term in $\delta\psi_\mu = 0$ is

$$-\frac{1}{8\kappa} \gamma_\mu \gamma_w (1 - \Gamma^4) \varepsilon_1 + \frac{1}{96} \gamma_\mu G \varepsilon_0^* = 0 . \quad (3.9)$$

From the structure of the equations we can assume that ε_1 (like ε_0^*) has the opposite chirality to ε_0 , namely $\Gamma^4 = -1$,³ and so this determines

$$\varepsilon_1 = \frac{\kappa}{24S^2} \gamma_w G \varepsilon_0^* . \quad (3.10)$$

We have defined

$$S^2 = \gamma_w \gamma_w = Z^{-1/2} w_m w_m . \quad (3.11)$$

³A term of the same chirality would have to be proportional to ε_0 and so can be absorbed in the latter.

The first order term in $\delta\psi_m = 0$ becomes

$$\partial_m \xi - \frac{1}{2} Z^{-1/2} G \gamma_m \eta^* = 0 , \quad (3.12)$$

where $\xi = Z^{-1/2} S^{-2} \gamma_w G \eta^*$. We have used the identity

$$\gamma_M^{PQR} G_{PQR} - 9 \gamma^{PQ} G_{MPQ} = -2G \gamma_M - \gamma_M G . \quad (3.13)$$

For $m = \bar{i}$, the property (3.5) immediately gives

$$\partial_{\bar{i}} \xi = 0 \quad (3.14)$$

so that ξ is a spinor holomorphic in the z^i and

$$G \eta^* = Z^{1/2} \gamma_w \xi(z) . \quad (3.15)$$

The final, $m = i$ equation, then becomes

$$G \gamma_i \eta^* = 2 Z^{1/2} \partial_i \xi(z) . \quad (3.16)$$

This can also be written with $i \rightarrow m$, as the \bar{i} components hold trivially.

We now wish to expand eqs. (3.15,3.16) in terms of the components of G . The most general holomorphic spinor ξ of the correct chirality is

$$\xi(z) = \alpha \eta^* + \frac{1}{2} \beta_{\bar{i}\bar{j}} \gamma^{\bar{i}\bar{j}} \eta^* , \quad (3.17)$$

where $\alpha(z)$ and

$$\hat{\beta}_{\bar{i}\bar{j}}(z) = Z^{-1/2} \beta_{\bar{i}\bar{j}} \quad (3.18)$$

are holomorphic. The factor of $Z^{-1/2}$ arises because it is the $\gamma^{\hat{m}}$, with tangent space index, that are matrices with constant coefficients, whereas

$$\gamma^m = Z^{-1/4} \gamma^{\hat{m}} \quad (3.19)$$

are position-dependent. Also,

$$\begin{aligned} G \eta^* &= G_{\bar{i}\bar{j}\bar{k}} \gamma^{\bar{i}\bar{j}\bar{k}} \eta^* + 3 G_{\bar{i}\bar{j}\bar{k}} \gamma^{\bar{i}\bar{j}\bar{k}} \eta^* \\ &= G_{\bar{i}\bar{j}\bar{k}} \gamma^{\bar{i}\bar{j}\bar{k}} \eta^* + 6 G_{\bar{i}\bar{j}\bar{k}} \gamma^{\bar{i}\bar{j}\bar{k}} \eta^* , \end{aligned} \quad (3.20)$$

and

$$\begin{aligned} G \gamma_i \eta^* &= 3 G_{\bar{l}\bar{j}\bar{k}} \gamma^{l\bar{j}\bar{k}} \gamma_i \eta^* + 3 G_{\bar{l}\bar{j}\bar{k}} \gamma^{l\bar{j}\bar{k}} \gamma_i \eta^* \\ &= 6 G_{\bar{i}\bar{j}\bar{k}} \gamma^{\bar{j}\bar{k}} \eta^* - 6 G_{\bar{j}\bar{k}}^{\bar{i}} \gamma^{\bar{i}\bar{k}} \eta^* + 12 G_{\bar{i}\bar{j}}^{\bar{j}} \eta^* . \end{aligned} \quad (3.21)$$

Expanding the condition (3.16) in components, we have

$$\partial_i \alpha = 6Z^{-1/2} G_{ij}{}^j = 0 , \quad (3.22)$$

where we have made use of the earlier condition (3.8). Thus, α is constant (which the Bianchi identities will require to vanish). For β , we have

$$Z \partial_i \hat{\beta}_{\bar{j}\bar{k}} = 6(G_{i\bar{j}\bar{k}} + G_{\bar{l}[\bar{j}}^{\bar{l}} g_{\bar{k}]i}) . \quad (3.23)$$

In order to analyze this condition it is useful to define

$$\begin{aligned} f_l(z) &= \frac{1}{2} \hat{\epsilon}_l{}^{\bar{j}\bar{k}} \hat{\beta}_{\bar{j}\bar{k}}(z) , \\ G_{il} &= \frac{1}{2} \hat{\epsilon}_l{}^{\bar{j}\bar{k}} G_{i\bar{j}\bar{k}} , \quad G_{\bar{l}} = \frac{1}{2} \hat{\epsilon}_l{}^{jk} G_{ijk} . \end{aligned} \quad (3.24)$$

Here $\hat{\epsilon}$ is the numerical ϵ -symbol, with constant values $0, \pm 1$ regardless of index positions. Then the condition (3.23) becomes

$$Z \partial_i f_l = 3(G_{il} + G_{li}) . \quad (3.25)$$

From the symmetry in i and l it follows that

$$f_l(z) = \partial_l \phi(z) , \quad \beta_{i\bar{j}}(z) = Z^{1/2} \hat{\epsilon}_{i\bar{j}}{}^l \partial_l \phi(z) \quad (3.26)$$

in terms of a general holomorphic function $\phi(z)$. Then the full content of condition (3.16) is

$$\partial_i \alpha(z) = 0 , \quad G_{il} + G_{li} = \frac{Z}{3} \partial_i \partial_l \phi(z) . \quad (3.27)$$

Similarly expanding the final condition (3.15), we find

$$\begin{aligned} G_{i\bar{j}\bar{k}} &= \frac{1}{6} \hat{\epsilon}_{i\bar{j}\bar{k}} \partial_{\bar{l}} Z \partial_l \phi , \\ G_{il} - G_{li} &= -\frac{1}{6} Z^{-1/2} (\alpha \partial_{\bar{j}} Z \hat{\epsilon}_{il}^{\bar{j}} + 4 \partial_{[i} \phi \partial_{l]} Z) . \end{aligned} \quad (3.28)$$

In summary, the conditions for $\mathcal{N} = 1$ supersymmetry are conveniently written by separating G_{il} and $G_{\bar{l}}$ into symmetric and antisymmetric parts,

$$G_{il} = S_{il} + A_{il} , \quad G_{\bar{l}} = S_{\bar{l}} + A_{\bar{l}} . \quad (3.29)$$

Then $S_{i\bar{l}}$ is completely undetermined, while eqs. (3.8, 3.27, 3.28) fix the rest in terms of one constant α and one holomorphic function ϕ :

$$\begin{aligned} G_{ijk} &= A_{i\bar{l}} = 0 , \\ S_{il} &= \frac{Z}{6} \partial_i \partial_l \phi , \\ A_{il} &= -\frac{1}{12} \alpha Z^{-1/2} \partial_{\bar{j}} Z \hat{\epsilon}_{il}^{\bar{j}} - \frac{1}{3} \partial_{[i} \phi \partial_{l]} Z , \\ G_{i\bar{j}\bar{k}} &= \frac{1}{6} \hat{\epsilon}_{i\bar{j}\bar{k}} \partial_{\bar{l}} Z \partial_l \phi . \end{aligned} \tag{3.30}$$

3.2 Bianchi identities and equations of motion

We now impose the Bianchi identity $dG_{(3)} = 0$ on the background. Expressed in terms of the fields (3.29), these take the form

$$\hat{\epsilon}^{\bar{j}\bar{k}\bar{l}} \partial_i G_{\bar{j}\bar{k}\bar{l}} = 6 \partial_{\bar{j}} G_{ij} \tag{3.31}$$

$$\partial_i G_{\bar{j}\bar{k}} - \partial_{\bar{j}} G_{i\bar{k}} = -\hat{\epsilon}^{ab}_{\bar{k}} \hat{\epsilon}^{ci}_{\bar{j}} \partial_a G_{bc} , \tag{3.32}$$

$$\hat{\epsilon}^{jkl} \partial_i G_{jkl} = 6 \partial_j G_{i\bar{j}} . \tag{3.33}$$

(We use a, b, c as well as i, j, k, l for holomorphic indices.) Eq. (3.31) reduces to

$$\partial_i \phi \partial_j \partial_{\bar{j}} Z = 0 . \tag{3.34}$$

Thus the Bianchi identity (3.31) holds except at the locations (2.10) of the D3-branes, where it should break down because these carry 5-brane charges as well in the PS solution.

Eqs. (3.32, 3.33) constrain the remaining components $G_{\bar{j}\bar{k}}$:

$$\partial_i G_{\bar{j}\bar{k}} - \partial_{\bar{j}} G_{i\bar{k}} = -\frac{1}{24} \alpha Z^{1/2} (\hat{\epsilon}_{\bar{k}\bar{i}\bar{j}} w_l w_{\bar{l}} - \hat{\epsilon}_{\bar{i}\bar{j}}^l w_l w_{\bar{k}}) - \frac{1}{6} \hat{\epsilon}^{ab}_{\bar{k}} \hat{\epsilon}^{ci}_{\bar{j}} \partial_b \phi \partial_a \partial_c Z , \tag{3.35}$$

$$\partial_i G_{i\bar{j}} = 0 . \tag{3.36}$$

Taking ∂_i of identity (3.35) and using identity (3.36) gives

$$\partial^2 G_{\bar{j}\bar{k}} = -\frac{1}{12} \alpha \partial_i \left[Z^{1/2} (\hat{\epsilon}_{\bar{k}\bar{i}\bar{j}} w_l w_{\bar{l}} - \hat{\epsilon}_{\bar{i}\bar{j}}^l w_l w_{\bar{k}}) \right] - \frac{1}{3} \hat{\epsilon}^{ab}_{\bar{k}} \hat{\epsilon}^{ci}_{\bar{j}} \partial_b \partial_i \phi \partial_a \partial_c Z ; \tag{3.37}$$

(note that $\partial^2 = 2\partial_i \partial_i$). Symmetry in $\bar{j}\bar{k}$ now implies that

$$\alpha = 0 . \tag{3.38}$$

Inverting eq. (3.37) gives

$$G_{\bar{j}\bar{k}} = -\frac{1}{3\partial^2} \epsilon^{ab} \hat{\epsilon}_{\bar{k}}^{ci} \partial_b \partial_i \phi \partial_a \partial_c Z . \quad (3.39)$$

To be precise, this final component is not completely determined, because eq. (3.37) allows us to add any harmonic tensor, subject to the earlier conditions $A_{\bar{j}\bar{k}} = \partial_j G_{\bar{j}\bar{k}} = 0$. The general solution is then

$$G_{\bar{j}\bar{k}} = -\frac{1}{3\partial^2} \epsilon^{ab} \hat{\epsilon}_{\bar{k}}^{ci} \partial_b \partial_i \phi \partial_a \partial_c Z + \partial_j \partial_{\bar{k}} \psi \quad (3.40)$$

for ψ any harmonic function.

In summary, the full solution to the supersymmetry conditions and Bianchi identities is given in eqs. (3.30) (with $\alpha = 0$) and (3.40) in terms of one holomorphic function ϕ and one harmonic function ψ . In the appendix we give the explicit form for $G_{(3)}$ in the special case where the cross derivatives of the holomorphic function ϕ are zero.

Now we will verify that all such solutions satisfy the equations of motion as well. To the order we are working, the only nontrivial equation of motion is that for $G_{(3)}$,

$$d*G_{(3)} + 4i\kappa G_{(3)} \wedge F_{(5)} = 0 . \quad (3.41)$$

In the 3-brane background, for transverse $G_{(3)}$, this becomes

$$d[Z^{-1}G_{(3)}^+] = 0 , \quad G_{(3)}^+ \equiv G_{(3)} + i*_6 G_{(3)} \quad (3.42)$$

where $*_6$ means the dual in the six-dimensional space with respect to the flat metric δ_{mn} .

Defining for $G_{(3)}^+$ the tensors G_{ij}^+ , $G_{\bar{i}\bar{j}}^+$, and their symmetric and antisymmetric parts, in parallel to the earlier definitions for $G_{(3)}^+$, on finds that

$$\begin{aligned} G_{\bar{i}\bar{j}\bar{k}}^+ &= A_{ij}^+ = S_{\bar{i}\bar{j}}^+ = 0 , \\ G_{ijk}^+ &= 2G_{ijk} , \quad A_{\bar{i}\bar{j}}^+ = 2A_{\bar{i}\bar{j}} , \quad S_{ij}^+ = 2S_{ij} . \end{aligned} \quad (3.43)$$

The only nontrivial component is then

$$S_{ij}^+ = \frac{Z}{3} \partial_i \partial_j \phi , \quad (3.44)$$

and the nontrivial equations of motion

$$\partial_{[i}(Z^{-1}S_{j]k}^+) = \partial_i(Z^{-1}S_{ij}^+) = 0 \quad (3.45)$$

are immediately seen to be satisfied.

In appendix A.1 we carry this to second order for the dilaton and axion, and in appendix A.2 we find the explicit form of $G_{\bar{i}\bar{j}}$ for special ϕ .

3.3 Discussion

On the gauge theory side, the perturbation studied in PS is, in $\mathcal{N} = 1$ notation, a mass term for the three chiral superfields. More generally, any $\mathcal{N} = 1$ superpotential would preserve one supersymmetry, and so it is natural to identify the holomorphic function ϕ with the superpotential. Let us check that the dimensions are correct, first for the case of pure AdS space where $Z = R^4/r^4$. Let ϕ be homogeneous of degree k , corresponding to a perturbation of dimension $\Delta = k + 1$. In this case all terms involving ϕ in the solution (3.30,3.40) for G_{mnp} scale as r^{k-6} , and all terms in the inertial frame $G_{\hat{m}\hat{n}\hat{p}}$ as $r^{k-3} = r^{\Delta-4}$. This is the correct scaling for the nonnormalizable solution dual to an operator of dimension Δ [5, 6], confirming the interpretation of ϕ as dual to a superpotential perturbation. We have verified that the supergravity solution for $\phi = mz^i z^i$ reproduces the nonnormalizable solution

$$G_{(3)} = r^{-4}(T_{(3)} - 4V_{(3)}/3) \quad (3.46)$$

where (in the notation of PS) the nonzero components of T are

$$T_{i\bar{j}\bar{k}} = m\hat{e}_{i\bar{j}\bar{k}} \ , \quad (3.47)$$

and $V_{(3)}/3$ is $T_{(3)}$ projecting out components orthogonal to x^m .

The PS solutions also have a normalizable part; with its inclusion the solution must still be supersymmetric. The solutions we have found here have no independent normalizable part; in particular, we will see that ψ corresponds to higher dimensional operators. Rather, the normalizable part is already determined in terms of ϕ and Z . With expanded 5-branes Z has terms subleading in $1/r$, and through the solution (3.30,3.40) these generate the normalizable part of $G_{(3)}$. In the appendix we verify that the component $G_{i\bar{j}\bar{k}}$ obtained here is in agreement with PS.

The harmonic function ψ produces solutions with the same $SO(6)$ quantum numbers as ϕ and with dimension Δ' greater by 4. This follows from eq. (3.40), where the two terms have the same net number of derivatives but the first has an extra factor of r^{-4} asymptotically from Z . Both branches appear in table 2 of ref. [29] (see also ref. [30]). Just as the superpotential perturbations correspond to operators of the form $\lambda^i \lambda^j \partial_i \partial_j \phi$, the solutions given by ψ have the dimensions and $SO(6)$ quantum numbers of the operators $F^2 \bar{\lambda}^i \bar{\lambda}^j \partial_i \partial_j \psi$. We have not fully understood from the field theory side why the latter are parameterized by a harmonic rather than holomorphic function.

4 A class of exact solutions

In this section we note an interesting class of exact solutions with $G_{(3)}$ flux. We begin with a Calabi-Yau background,

$$\widetilde{ds^2} = \eta_{\mu\nu} dx^\mu dx^\nu + \widetilde{ds_K^2} , \quad (4.1)$$

with $\widetilde{ds_K^2}$ is a Ricci-flat metric on the transverse space K . The dilaton-axion field is constant,

$$\tau = \frac{\theta}{2\pi} + \frac{i}{g} , \quad (4.2)$$

and the other IIB supergravity fields vanish,

$$F_{(3)} = H_{(3)} = F_{(5)} = 0 . \quad (4.3)$$

The dilatino variation vanishes trivially, while the gravitino variation vanishes for

$$\partial_\mu \tilde{\varepsilon} = \tilde{D}_m \tilde{\varepsilon} = 0 \quad (4.4)$$

Thus there are two $D = 4$ supersymmetries (from the real and imaginary parts of ε) for each covariantly constant spinor on K .

Now introduce a warp factor $Z(x^m)$,

$$ds^2 = Z^{-1/2} \eta_{\mu\nu} dx^\mu dx^\nu + Z^{1/2} \widetilde{ds_K^2} \quad (4.5)$$

and a five-form flux (2.8),

$$F_5 = d\chi_4 + *d\chi_4 , \quad \chi_4 = \frac{1}{4\kappa Z} dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 , \quad (4.6)$$

with constant dilaton-axion and vanishing three-form fluxes. The dilatino variation is zero, while

$$\begin{aligned} \kappa \delta \psi_\mu &= \partial_\mu \varepsilon - \frac{1}{8} \gamma_\mu \gamma_w (1 - \Gamma^4) \varepsilon , \\ \kappa \delta \psi_m &= \tilde{D}_m \varepsilon + \frac{1}{8} \varepsilon w_m - \frac{1}{8} \gamma_w \gamma_m (1 - \Gamma^4) \varepsilon . \end{aligned} \quad (4.7)$$

There is then an unbroken supersymmetry [15]

$$\varepsilon = Z^{-1/8} \tilde{\varepsilon} \quad (4.8)$$

for each covariantly constant spinor of chirality $\Gamma^4 \tilde{\varepsilon} = \tilde{\varepsilon}$. This is the familiar multi-three-brane metric: it is a sourceless solution to the equations of motion for Z a harmonic function of the transverse coordinates, and more generally is a solution with D3-brane sources

$$-\tilde{\nabla}_K^2 Z = (2\pi)^4 \alpha'^2 g \rho_3(x^m) . \quad (4.9)$$

Now we construct a solution with nonzero $G_{(3)}$. We need on K a 3-form $\omega_{(3)}$ which is both closed and divergenceless,

$$d\omega_{(3)} = d*_K\omega_{(3)} = 0 . \quad (4.10)$$

A harmonic 3-form on a compact manifold, or with sufficiently rapid falloff on a noncompact manifold, will have this property. By forming linear combinations (and taking a complex conjugate if needed) we may assume that

$$*_K\omega_{(3)} = i\omega_{(3)} \quad (4.11)$$

Then $G_{(3)} = C\omega_{(3)}$ solves the equations of motion, where the metric, dilaton-axion, and 5-form are still of black 3-brane form. The $G_{(3)}$ equation of motion (3.42) is trivial. The field equation for the metric, at constant τ , is

$$R_{MN} - \frac{\kappa^2}{6} F_{MPQRS} F_N{}^{PQRS} = \frac{\kappa^2}{4} G_{(M|PQ} G_{|N)}{}^{PQ*} - \frac{\kappa^2}{48} g_{MN} G_{PQR} G^{PQR*} . \quad (4.12)$$

For the black 3-brane Ansatz, the left-hand side is

$$L_{\mu\nu} = \eta_{\mu\nu} \frac{1}{4Z^2} \tilde{\nabla}^2 Z , \quad L_{mn} = -\tilde{g}_{mn} \frac{1}{4Z} \tilde{\nabla}^2 Z . \quad (4.13)$$

On the right-hand side, the condition (4.11) implies that

$$\begin{aligned} G_{mpq} G_n{}^{pq*} &= \frac{1}{36} \epsilon_{mpq}{}^{rst} \epsilon_n{}^{pquvw} G_{rst} G_{uvw}^* \\ &= \frac{1}{3} g_{mn} G_{rst} G^{rst*} - G_{npq} G_m{}^{pq*} , \end{aligned} \quad (4.14)$$

and so

$$G_{(m|pq} G_{|n)}{}^{pq*} = \frac{1}{6} g_{mn} G_{pqr} G^{pqr*} . \quad (4.15)$$

The field equation is then satisfied for

$$-\tilde{\nabla}^2 Z = (2\pi)^4 \alpha'^2 g \rho_3(x^m) + \frac{\kappa^2}{12} G_{pqr} \widetilde{G^{pqr*}} . \quad (4.16)$$

Note that it is the original, tilded, metric on K that appears here. The Bianchi identity,

$$dF_{(5)} = -4i\kappa G_{(3)} \wedge G_{(3)}^* , \quad (4.17)$$

is satisfied under the same condition (4.16).

Not all these solutions are supersymmetric, but they become so if we impose the additional condition that $\omega_{(3)}$ contain only $(2,1)$ and $(1,2)$ components under the complex structure defined by the supersymmetry of the Calabi-Yau manifold K . In other words,

$$\omega_{(3,0)} = \omega_{(0,3)} = 0 . \quad (4.18)$$

To see this, note first that the self-duality condition (4.11) implies in the notation of eq. (3.43) that $\omega^+ = 0$ or

$$\omega_{ijk} = \alpha_{\bar{i}\bar{j}} = \sigma_{ij} = 0 , \quad (4.19)$$

where σ and α are the symmetric and antisymmetric parts of ω , defined by analogy to the earlier S and A . The $(0,3)$ condition (4.18) implies that also $\omega_{\bar{i}\bar{j}\bar{k}} = 0$. Finally, $\omega_{ij}{}^{\bar{j}}$ must vanish, else it would be a harmonic $(1,0)$ form, which does not exist on a Calabi-Yau manifold. This is equivalent to $\alpha_{ij} = 0$. It follows that the only nontrivial component of $\omega_{(3)}$ is $\sigma_{\bar{i}\bar{j}}$. Now we must consider the supersymmetry conditions, treating ε exactly. We claim that the fermionic fields remain invariant for the same spinor (4.8) as in the absence of $G_{(3)}$. The terms that do not involve $G_{(3)}$ in the variations (2.1,2.2) already vanish, so those that involve $G_{(3)}$ must vanish separately. By a calculation directly parallel to that which led to eq. (3.30), one sees that the nonzero components $\sigma_{\bar{i}\bar{j}}$ do not appear in the variations, which therefore vanish.

Note the structure of this solution, with its strong resemblance to the D3-brane Higgs branch solution. The flux $G_{(3)} \wedge G_{(3)}^*$ behaves like an additional D3-brane density, but without the moduli of D3-branes — perhaps one can think of this flux as a sort of frozen density of D3-branes. In fact, this freezing seems to be a manifestation of confinement: taking K to be the deformed conifold, this is precisely the recent KS solution [21]. For the conifold itself, we obtain the solution of Klebanov and Tseytlin [20], which is singular because the integrated D3-brane density diverges; the resolved conifold would lead to a similar singular solution. For $\mathbf{R}^2 \times \mathbf{R}^4/\mathbf{Z}_2$ one obtains an $\mathcal{N} = 2$ analog of the KS solution [22].

The solution found in this section resembles the solution found by Becker and Becker for M theory on a Calabi-Yau four-fold [23]. In fact, it is a special case, if one takes the four-fold to be a three-fold times T^2 , and then takes the T -dual on T^2 as in refs. [24, 25] to obtain a IIB solution.

5 Conclusions

We have verified the supersymmetry of the PS solution to first order in the perturbation, and of its generalization to an arbitrary $\mathcal{N} = 1$ superpotential, and we have shown that this

condition together with the Bianchi identity determines the solution. The $\mathcal{N} = 1$ conditions may be a useful method to find the exact solution, and so describe physics that is outside the approximation used in PS.

We have also found an interesting exact solution, which includes the KS solution but not the PS solution — the three-form flux in the latter case is not of the form $*_K G_{(3)} = iG_{(3)}$, and the dilaton is not constant. It would be useful to find a generalization which includes both solutions (and also the solution [31]), and so obtain a more universal understanding of the supergravity duals of confining theories. The more general Becker–Becker solutions [23, 24, 25] may be useful here.⁴ It is a further useful direction to incorporate these noncompact solutions as local regions in a compactified space (as in [4]), and so produce Randall–Sundrum type compactifications with large warp factors and four-dimensional gravity.

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A Appendix

A.1 Second order dilaton and axion

At second order in the expansion there is a nonconstant dilaton and axion and a correction to the metric and $F_{(5)}$. These have recently been obtained directly from the equations of motion by Freedman and Minahan [33], who also considered the finite temperature case. Here we will simply verify that the supersymmetry equations determine the second order (zero temperature) dilaton and axion. The second order dilatino variation involves only these second order corrections,

$$\frac{i}{\kappa} \gamma^M P_M \varepsilon_0^* - \frac{i}{24} \gamma^{MNP} G_{MNP} \varepsilon_1 = 0 . \quad (\text{A.1})$$

Inserting the first order solution, this becomes

$$P_{\bar{i}} \gamma^{\bar{i}} \eta^* = \frac{\kappa^2}{24^2 S^2} G \tilde{\gamma} G \eta^* = \frac{\kappa^2}{24} G_{\bar{i}\bar{j}} \partial_j \phi \gamma^{\bar{i}} \eta^* . \quad (\text{A.2})$$

⁴The relevance of these solutions to Randall–Sundrum compactification has recently been discussed in refs. [32].

Expanding $B = B_0 + \delta B$, this is

$$f_0^2 \partial_{\bar{i}} \delta B = \frac{\kappa^2}{24} G_{\bar{i}j} \partial_j \phi , \quad (\text{A.3})$$

which is integrable by eq. (3.35):

$$\delta B = \frac{\kappa^2}{12 f_0^2} \frac{1}{\partial^2} G_{\bar{i}j} \partial_i \partial_j \phi . \quad (\text{A.4})$$

The condition (A.3) allows an arbitrary holomorphic piece $F(z)$ in δB . This corresponds to an additional $\mathcal{N} = 1$ interaction

$$\int d^2\theta F(\Phi) W_\alpha W^\alpha . \quad (\text{A.5})$$

A.2 Particular solution

In eqs. (3.40) and (A.4) we have given the solutions in terms of the Green function ∂^{-2} . Here we note that for $Z = R^4/r^4$ and ϕ of the form

$$\phi = \sum_{i=1}^3 f_i(z^i) \quad (\text{A.6})$$

(which includes the mass term $\frac{1}{2} m z^i z^i$ as a special case) we can give a closed form for each. Specifically,

$$G_{\bar{1}\bar{1}} = \frac{2R^4}{3r^6} \left[\bar{z}^2 \bar{z}^2 \frac{\partial_3 \phi}{z^3} + \bar{z}^3 \bar{z}^3 \frac{\partial_2 \phi}{z^2} \right] + \partial_{\bar{1}} \partial_{\bar{1}} \psi \quad (\text{A.7})$$

(and the same permuted for the other diagonal terms) and

$$G_{\bar{1}\bar{2}} = \frac{2R^4}{3r^6} \bar{z}^1 \bar{z}^2 \frac{\partial_3 \phi}{z^3} + \partial_{\bar{1}} \partial_{\bar{2}} \psi \quad (\text{A.8})$$

and permutations for the off-diagonal terms. For the second order dilaton and axion, we obtain:

$$\delta B = -\frac{\kappa^2 R^4}{144 f_0^2 r^4} \left(\bar{z}^1 \bar{z}^1 \frac{\partial_2 \phi \partial_3 \phi}{z^2 z^3} + \bar{z}^2 \bar{z}^2 \frac{\partial_1 \phi \partial_3 \phi}{z^1 z^3} + \bar{z}^3 \bar{z}^3 \frac{\partial_1 \phi \partial_2 \phi}{z^1 z^2} \right) + \frac{\kappa^2}{24 f_0^2} \partial_i \phi \partial_{\bar{i}} \psi + H(z) , \quad (\text{A.9})$$

where H is any holomorphic function.

A.3 Comparison to PS

In this appendix we compare our solution to that of PS, in particular to verify that the normalizable part arises as argued in section 3.3. We focus on the solution with a single D5-sphere. The PS solution was

$$G_{(3)} = *_6 d\omega_{(2)} + id\omega_{(2)} + d\eta_{(2)} , \quad (\text{A.10})$$

where $\eta_{(2)}$ is the background field, while the $\omega_{(2)}$ terms are from the brane source.

The potentials are

$$\begin{aligned} \omega_{(2)} &= -\frac{\alpha'}{4w^3} \epsilon_{ijk} w^i dw^j \wedge dw^k \left(-\ln \frac{A}{B} + \frac{2r_0 w}{A} + \frac{2r_0 w}{B} \right) , \\ \eta_{(2)} &= \frac{\alpha'}{2\sqrt{2}mw^3} T_{mnp} dx^n \wedge dx^p \\ &\quad \times \left(-ww_{,m} \ln \frac{A}{B} + 2(w+r_0)w^2 w_{,m} \left[\frac{1}{A} + \frac{1}{B} \right] + 2w^2 y y_{,m} \left[\frac{1}{A} + \frac{1}{B} \right] \right) , \end{aligned} \quad (\text{A.11})$$

where $r_0 = \pi\alpha' mN$ and

$$\begin{aligned} y^i &= \frac{z^i + \bar{z}^i}{\sqrt{2}} , \quad w^i = i \frac{z^i - \bar{z}^i}{\sqrt{2}} , \\ A &= y^2 + (w+r_0)^2 , \quad B = y^2 + (w-r_0)^2 . \end{aligned} \quad (\text{A.12})$$

This does indeed agree with the result in section 3. Consider for example the component $G_{\bar{1}\bar{2}\bar{3}}$, for which the earlier result was

$$G_{\bar{1}\bar{2}\bar{3}} = \frac{1}{6} \partial_{\bar{l}} Z \partial_{\bar{l}} \phi , \quad (\text{A.13})$$

corresponding to the potential ($G_{(3)} = d\Lambda_{(2)}$)

$$\Lambda_{\bar{j}\bar{k}} = \frac{1}{6} \hat{\epsilon}^i_{\bar{j}\bar{k}} Z \partial_i \phi . \quad (\text{A.14})$$

For the $\bar{1}\bar{2}\bar{3}$ component, $*_6 d\omega_2 = id\omega_2$ and so the PS solution takes the form

$$G_{\bar{1}\bar{2}\bar{3}} = d(\eta + 2i\omega)_{\bar{1}\bar{2}\bar{3}} . \quad (\text{A.15})$$

For the solution (A.11) we find

$$(\eta + 2i\omega)_{\bar{j}\bar{k}} = \frac{mN\alpha'^2 \sqrt{2}}{AB} z^i \hat{\epsilon}^i_{\bar{j}\bar{k}} . \quad (\text{A.16})$$

We should note the translation between Schwarz's conventions, used here, and the conventions in PS:

$$F_{(5)}^{\text{PS}} = \frac{4\kappa}{g} F_{(5)}^{\text{S}} , \quad G_{(3)}^{\text{S}} = \frac{\kappa}{g} G_{(3)}^{\text{S}} . \quad (\text{A.17})$$

(The general normalization for $G_{(3)}^{\text{S}}$ beyond linear order is more complicated; also we have assumed for convenience that $\theta = 0$.) Noting also that $Z = R^4/AB$, the present (A.14) and PS (A.16) results agree for $\phi = 3\sqrt{2}(g/\kappa)mz^iz^i$, which is indeed the superpotential up to a numerical constant.

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